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The Discrete Laplace Operator in an Octant

A.I. OVSEEVICH

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1 Introduction

This paper is an attempt to extend to higher dimensions some methods and results pertaining to one- and two-dimensional Toeplitz operators (see [1]). To this end we consider a very special multi-dimensional Toeplitz operator, namely, the discrete Laplacian. Here, we leave aside possible physical and probabilistic applications. Our main results are an explicit inversion formula for the Laplacian based upon the Bergman - Weil integral representation and the analytic continuation of the resolvent to a multi-sheeted Riemann surface. Besides, we develop some homological mashinery, thus providing an alternative approach to the above problems. As a whole the paper provides yet another evidence of might of homological and algebraic-geometrical methods. It seems plausible that, conversely, Toeplitz operators can play an important role in geometry. Some indications of this kind provides Tate's approach to residues and duality in algebraic geometry. It is tempting to suspect that by means of Toeplitz operators one can achieve fruitful interaction of the Connes noncommutative geometry with the usual commutative one.

2 Laplacian inversion and decomposition of holomorphic functions

Let $Op(\varphi)$ be the Toeplitz operator related to a function φ (symbol of the operator). The function φ is continuous and defined upon the torus $T^n = \{z \in \mathbf{C}^n, |z_i| = 1\}$, while the operator $Op(\varphi)$ acts upon the Hardy space of analytic functions $H^2(D^n)$ (where $D^n = \{z \in \mathbf{C}^n, |z_i| < 1\}$ is the unit polydisk) via the composition $f \mapsto f\varphi \mapsto Pf\varphi$ where P is the standard orthogonal projection $L^2(D^n) \to H^2(D^n)$. We are interested in the operator $Op(\sum_{i=1}^n z_i + z_i^{-1} - \lambda)$ or rather its inverse $R_\lambda = Op(\sum_{i=1}^n z_i + z_i^{-1} - \lambda)^{-1} = Op(\varphi_\lambda)^{-1}$. It is clear that the equality $R_\lambda f = u$, where $f, u \in H^2(D^n)$ means exactly that

$$\varphi_\lambda u = f + \sum z_i^{-1} f_i,$$

where f_i does not depend on $z_i, f_i \in H^2(D^n)$. According to the Hilbert Nullstellensatz or rather its analytic version this equation is equivalent to

$$z_1 \ldots z_n f(z) = -\sum (z_1 \ldots \hat{z}_i \ldots z_n) f_i(z_1, \ldots, \hat{z}_i, \ldots z_n) \qquad (2.1)$$

for $z \in V_\lambda \cap D^n$, where

$$V_\lambda = \{z \in \mathbf{C}^n, z_1 \dots z_n(\sum_{i=1}^{n}(z_i + z_i^{-1}) - \lambda) = 0\}.$$

In other words we have to solve the following problem. Let

$$V^* = \{z \in \mathbf{C}^n, z_1 \dots z_n(\sum_{i=1}^{n} z_i + z_i^{-1} - \lambda) = 0, |z_i| < 1\}$$

and let $g(z)(= z_1 \dots z_n f(z))$ be a holomorphic function on V^* which vanishes as $z_i = 0$ for some i. We must represent g as a sum

$$g(z) = \sum g_i(z), \tag{2.2}$$

where g_i is holomorphic on V^*, does not depend on z_i and vanishes as soon as $z_j = 0, \forall j \neq i$. By using new coordinates $u_i = z_i + z_i^{-1}$ we arrive at the following problem. Let $\Omega = \{u \in \mathbf{C}^n, \sum_{i=1}^{n} u_i = \lambda, \forall i, u_i \notin [-2, 2]\}$ (it is an isomorphic image of the analytic set V^* with coordinate hyperplanes $z_i = 0$ deleted), f is an analytic function on Ω that tends to zero as $u \to \infty$ (that is as some $z_i \to 0$). One needs to expand f into a sum $f(u) = \sum f_i(u_1, \dots, \widehat{u}_i, \dots, u_n)$ wheref_i are holomorphic in Ω, do not depend on u_i and, moreover, decrease at infinity.

3 Integral representation

A solution can be obtained by means of the (slightly generalized) Bergman-Weil representation.The only problem is that this representation can not be applied directly to unbounded domains. Here we overcome this difficulty by using decreasing of the funct ion f at infinity. The result takes the following form.

Theorem 1 *Let $z \in \Omega$. Then*

$$\begin{aligned} f(z) &= \sum f_i & (3.1)\\ f_i(u_1, \dots, \widehat{u}_i, \dots, u_n) &= (\frac{1}{2\pi i})^{n-1} \int_{\Gamma_i} f(\zeta)\varphi_i, & (3.2) \end{aligned}$$

where

$$\begin{aligned} \varphi_i &= d\log(\zeta_1 - u_1) \wedge \dots \wedge d\log\widehat{(\zeta_i - u_i)} \wedge \dots \wedge d\log(\zeta_n - u_n) & (3.3)\\ \Gamma_i &= \{\zeta \in \bar{\Omega}, \zeta_j \in [-2, 2], j \neq i\} & (3.4) \end{aligned}$$

Here, the cycles Γ_i should not intersect each other, or what is the same $\lambda \notin [-2n, 2n]$. It is clear that $u_i \to \infty$ as $z_i \to 0$ and the differential form $d\log(\zeta_i - u_i)$ tends to zero. Therefore, the function s f_k do in fact vanish as $z_i = 0, \forall i \neq k$.

4 Proof of the integral representation

We sketch here a proof of the integral representation (3.2) for $n = 3$ case, following the classical paper[2] by A.Weil.

Theorem 2 *Let $\Omega = \{u \in \mathbf{C}^n, u_1 + u_2 + u_3 = 0, u_i \in D_i\}$. D_i being a smooth bounded domain in $\mathbf{C}$ with boundary K_i, $f \in \mathcal{O}(\Omega)$ be a holomorphic function in $\bar{\Omega}$ .Suppose that $K_1 \times K_2 \times K_3$ does not intersect with $a r\Omega$. Then, if $z \in \Omega$ we have*

$$f(z) = \frac{1}{(2\pi i)^2} \sum_{i<j} \int_{\sigma_{ij}} f(u) d\log(u_i - z_i) \wedge d\log(u_j - z_j) \tag{4.1}$$

$$\sigma_{ij} = \{\zeta \in \bar{\Omega}, z_i \in K_i, z_j \in K_j\} \tag{4.2}$$

Proof. Put $\varphi_i(u) = d\log(u_i - z_i), \varphi_{ij} = \varphi_i \wedge \varphi_j$. Then we have *identites fondamentales* of A.Weil

$$\varphi_{ij} + \varphi_{ji} = 0, \varphi_{ij} + \varphi_{jk} + \varphi_{ki} = 0$$

or, in other words, we get a Cech cocycle.Denote by $\mathcal{J}$ the right hand side of (4.1). We claim that if $z \notin \Omega$ then $\mathcal{J} = 0$. In fact,without loss of generality we may and do suppose that $z_1 \notin D_1$, while $z_i \notin K_i, i \neq 1$.Then, due to Weil-Cech relations we have

$$\varphi_{ij} = \varphi_{1i} - \varphi_{1j}.$$

On the other hand we have for

$$\sigma_i = \partial_i \Omega = \{z \in \Omega, z_i \in K_I\}$$

the followng geometric identity $\partial \sigma_i = \sum_j \sigma_{ij}$.Moreover, the form φ_{1i} is regular on σ_i. Therefore,

$$\sum_j \int_{\sigma_{ij}} \varphi_{1i} f = \int_{\partial \sigma_i} \varphi_{1i} f = \int_{\sigma_i} d(\varphi_{1i} f) = 0.$$

Here the second equality is just the Stocks theorem and the latter equality follows from the fact that $\varphi_{1i}f$ is a holomorphic form of higher degree and is, therefore, closed. Now by summing the identities just obtained and again taking Weil-Cech into account we have $\mathcal{J} = 0$. So our claim is proved.

By using it we deform our domain Ω into a polydisk, thus reducing everything to the Cauchy formula. More precisely, let $z \in \Omega$. Consider $\mathcal{J} = \mathcal{J}(D_1, \ldots, D_3)$as a function of the domains D_i. Let all domains D_i save for D_1 be fixed and we write simply $\mathcal{J} = \mathcal{J}(D_1)$.It is clear that the function $D \mapsto \mathcal{J}(D)$ is additive:$\mathcal{J}(D) = \mathcal{J}(D') + \mathcal{J}(D'')$ if D is a disjoint union of D' and D''. Now if we take a small polydisk $\{|z_1| < \varepsilon, |z_2| < \varepsilon\}$ for D', then $\mathcal{J}(D) = \mathcal{J}(D')$ because of our previous claim . It remains to note that the case $D_i, i = 1, 2, D_3 = \mathbf{C}$ (strictly speaking D_3 should be taken as a disk B_R with sufficiently large radius R) coinc ides with the (two-dimensional) Cauchy formula.

The integral representation (3.2) can now be obtained by applying the above theorem to the following situation:

$$D_i = B_R \setminus [-2, 2]_\varepsilon,$$

where X_ε stands for $\varepsilon-$neighbourhood of X. Then, the integrals over "outer" bo undary ∂B_R tend to the zero as $R \to \infty$ because of vanishing of the function at infinity, while the remaining part of the integral (4.1) converges to the integral (3.2) as $\varepsilon \to \infty$.

5 Homological approach

Here we describe another, more abstract approach to above problems. It enables us to obtain existence and uniqueness results concerning inversion of some Toeplitz operators.However, at present, this approach does not provide sufficiently powerful tool fo r analytic continuation problems.

We begin with introducing some notations. Let $\bar{V}$ be the closure of the surface $\{\sum z_i + z_i^{-1} = \lambda\}$ in $\underbrace{\mathbf{P}^1 \times \ldots \mathbf{P}^1}_{n}$. There is a group $G = (\mathbf{Z}/2)^n$ generated by transformations $\varepsilon_i : z_i \mapsto z_i^{-1}, z_j \mapsto z_j, \forall j \neq i$ which acts on $\bar{V}$. Put $U_i = \{z \in \bar{V}, |z_i| < 1\}$ and $U_{i,\varepsilon} = \varepsilon(U_i)$ for $\varepsilon \in G$. Then $\mathcal{U} = \{U_{i,\varepsilon}\}$ is a G-invariant open covering of the space $\bar{V}$. Here we suppose as above that $\lambda \notin [-2n, 2n]$ or, what is the same,that $\bar{V}$ does not intersect with the torus $T^n = \{z \in \mathbf{C}^n, |z_i| = 1 \forall i\}$. Consider the Cech complex $C^{\cdot}$ associated to the covering $\mathcal{U}$ and the structural sheaf $\mathcal{O}_{\bar{V}}$ and its subcomplex $\tilde{C}^{\cdot}$ of

G-invariant cochains. Recall that $C^{\cdot}$ is the sequence of abelian groups C^q equipped with differentials $\delta = \delta_q : C^q \to C^{q+1}$ defined as follows:

$$C^q = \oplus_{(i_0 \ldots i_q)} \Gamma(U_{i_0 \ldots i_q}, \mathcal{O}),$$

where

$$U_{i_0 \ldots i_q} = U_{i_0} \cap \ldots \cap U_{i_q}$$

and Γ stands for the space of sections (of the structural sheaf).

$$(\delta_q \varphi)_{i_0 \ldots i_{q+1}} = \sum (-1)^k \varphi_{i_0 \ldots \widehat{i_k} \ldots i_{q+1}},$$

where $\varphi_{j_0 \ldots j_q} \in \Gamma(U_{j_0 \ldots j_q}, \mathcal{O})$ are components of the cochain φ.

It is not hard to understand that vanishing of homologies of the complex $\hat{C}^{\cdot}$ means exactly that some decomposition problem for holomorphic functions admits a solution. For instance, if $H^{n-1}(\hat{C}^{\cdot}) = 0$ then the decomposition problem

$$g = \sum g_i,$$

where g is a holomorphic function on $V^* = U_{1,2,\ldots,n,id} = U_{1,id} \cap \ldots \cap U_{n,id}$ and g_i are also holomorphic on V^* and do not depend on z_i is solvable. Similarly, if $H^{n-2}(\hat{C}^{\cdot}) = 0$ then if $\sum f_i = 0$, where f_i are holomorphic on V^* and do not depend on z_i, these f_i can be represented as

$$f_i = \sum_{j \neq i} (-1)^i g_{ij},$$

where g_{ij} are holomorphic on V^* and do not depend on z_i, z_j.

We show here, following Grothendieck's ideas [3], that $H^i(\hat{C}^{\cdot}) = 0$ as $i > 0$. In fact:

a) $H^i(\hat{C}^{\cdot}) = H^i(C^{\cdot})^G = H^i(\bar{V}, \mathcal{O}_{\bar{V}})^G$, since the covering $\mathcal{U}$ turns out to be acyclic with respect to cohomologies of the G-sheaf $\mathcal{O}$.

b) the cohomology group $H^i(\bar{V}, \mathcal{O}_{\bar{V}})^G$ can be calculated at least in two ways: either by direct calculation of $H^i(\bar{V}, \mathcal{O}_{\bar{V}})$ with G-action, or by equality $H^i(\bar{V}, \mathcal{O}_{\bar{V}})^G = H^i(\bar{V}/G, \mathcal{O}_{\bar{V}/G})$. It is not hard to see that $\bar{V}/G = \mathbf{P}^1 \times \mathbf{P}^1$ and we are done. Now we state explicitly some consequences of this cohomology vanishing .

Corollary 1 *The Toeplitz operator $Op(\sum_{i=1}^{i=n} z_i + z_i^{-1} - \lambda)$ has right inverse as soon as $\lambda \notin [-2n, 2n]$.*

Corollary 2 *Under the same assumption the Toeplitz operator*

$$Op(\sum_{i=1}^{i=n} z_i + z_i^{-1} - \lambda)$$

has zero kernel.

We sketch here a proof of this second corollary for the simplest $n = 3$ case .

What we have to prove is that if $\sum_{i=1}^{i=n} f_i = 0$, where f_i is holomorphic on V^* does not depend on z_i and vanishes as soon as $z_j = 0, j \neq i$, then $f_i \equiv 0$.

In fact, according to above considerations, we obtain

$$f_i(z_j, z_k) = g_j(z_j) - g_k(z_k),$$

where g_j are analytic in V^* and depend only on z_j. (However, unlike f_i these functions g_j do not necessary vanish if their argument vanishes) Now, if we put $z_j = 0$ in the preceding equality we obtain $0 = g_j(0) - g_k(z_k)$. Thus, the functions g_k are in fact constant and , moreover, do not depend on$k$. It means just that $f_i \equiv 0$.

6 Matrix elements of the resolvent

Let $f, g \in H^2(D^n)$. We define a scalar product $< f, g >$ by

$$< f, g >= (\frac{1}{2\pi i})^n \int_{T^n} f(z_1, \ldots, z_n) g(z_1^{-1}, \ldots, z_n^{-1}) \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}$$

which differs from the scalar product in H^2 by transformation $g \mapsto \bar{g}$, where $\bar{g}(z) = \overline{g(\bar{z})}$ and the bar in the right hand side of the last equality stands for complex conjugation. Thus, $< f, g >= (f, \bar{g})$, where $(\cdot, \cdot)$ stands for the standard scalar product in $H^2(D^n)$. We prefer using $< \cdot, \cdot >$ instead of $(\cdot, \cdot)$ since the former is "more algebraic".

In the rest of the section we suppose that f and g are polynomials. According to section 1 $R_\lambda f$ can be written out as follows:

$$R_\lambda f(z) = \frac{1}{P_\lambda(z)}(z_1 \ldots z_n f(z) - \sum f_i(z_1, \ldots, \hat{z}_i, \ldots z_n). \qquad (6.1)$$

where $P_\lambda(z) = z_1 \ldots z_n(\sum(z_i + z_i^{-1}) - \lambda)$,

$$f_i(z_1, \ldots, \hat{z}_i, \ldots, z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{\Gamma_i} \zeta_1 \ldots \zeta_n f(\zeta) \varphi_i, \qquad (6.2)$$

where

$$\varphi_i = d\log(u_1 - v_1)\wedge\ldots\wedge d\log\widehat{(u_i - v_i)}\wedge\ldots\wedge d\log(u_n - v_n) \quad (6.3)$$
$$u_i = \zeta_i + \zeta_i^{-1}, v_i = z_i + z_i^{-1} \quad (6.4)$$
$$\Gamma_i = \{(\zeta_1,\ldots,\zeta_n) = \zeta \in \mathbf{C}^n, P_\lambda(\zeta) = 0, |\zeta_j| = 1, j \neq i, |\zeta_i| < 1\}. \quad (6.5)$$

Thus, $< R_\lambda f, g >$ can be rewritten in the form

$$< R_\lambda f, g > = \frac{1}{(2\pi i)^n}\int_{T^n}\frac{f(x)g(x^{-1})}{P_\lambda(x)}dx - \frac{1}{(2\pi i)^n}\sum_{i=1}^{n}\int_{T^n}\frac{f_i(x)g(x^{-1})}{P_\lambda(x)}\frac{dx}{x} =$$

$$= I(\lambda) - \sum_{i=1}^{n} I_i(\lambda).$$

The integral $I_i(\lambda)$,in turn, can be rewritten as an integral over a cycle on the surface $\{P_\lambda = 0\}$ due to the fact that $f_i(x)$ does not depend on x_i. Namely, by using the residue formula we obtain for $|x_j| = 1, \forall j \neq i$

$$\frac{1}{2\pi i}\int_{|x_i|=1}\frac{g(x_1^{-1},\ldots,x_i^{-1},\ldots,x_n^{-1})}{P_\lambda(x)}\frac{dx_i}{x_i} = -\frac{g(x^{-1})}{x_i(\partial P_\lambda/\partial x_i)(x)} \quad (6.6)$$

where $x = (x_1,\ldots,x_i,\ldots,x_n)$ lies on the surface $V_\lambda = \{P_\lambda = 0\}$ and $|x_i| > 1, |x_j| = 1 \forall j \neq i$.

The minus sign in (6.6) is due to the fact that x_i lies outside the circle $|x_i| = 1$. To explain why only one summand is present in the right hand side of (6.6) we recall that

a) the equation $P_\lambda = 0$ has a single root $x_i, |x_i| > 1$ for fixed $x_j, j \neq i$.

b) $g(x)$ is a polynomial and,therefore, $g(x^{-1})$ is nonsingular if $|x_j| \geq 1, \forall j$.

The formula (6.6) can be rewritten in a more invariant form

$$\frac{1}{2\pi i}\int_{|x_i|=1}\frac{g(x^{-1})}{P_\lambda(x)}\frac{dx_i}{x_i} = -\frac{g(x^{-1})}{x_i}\frac{\omega_\lambda(x)}{dx_1\ldots\widehat{dx_i}\ldots dx_n}, \quad (6.7)$$

where

$$\omega_\lambda = \frac{dx_1\ldots\ldots dx_n}{dP_\lambda(x)}$$

is the canonical differential of higher degree on the surface V_λ. We obtain from (6.7) that

$$I_i(\lambda) = -(\frac{1}{2\pi i})^{n-1}\int_{\Sigma_i(\lambda)}\frac{f_i(x)g(x^{-1})}{x}\omega_\lambda(x). \quad (6.8)$$

where x outside brackets is abbreviation for $x_1 \ldots x_n$ and

$$\Sigma_i = \Sigma_i(\lambda) = \{x \in V_\lambda, |x_j| = 1-, \forall j \neq i, |x_i| > 1\}.$$

Here, $1-$ means $1-\varepsilon$ with sufficiently small $\varepsilon > 0$. We replace 1 by $1-\varepsilon$ because the Bergman-Weil representation for $f_i(x)$ becomes singular as x approaches the boundary of the domain $V^* \subset V_\lambda$. When substituting (6.2),(6.3) for f_i in (6.8) we obtain

$$I_i(\lambda) = -(\frac{1}{2\pi i})^{2n-1} \int_{\Sigma_i \times \Gamma_i} f(\zeta)\zeta_1 \ldots \zeta_n g(x^{-1}) x_1^{-1} \ldots x_n^{-1} \Omega_i(\zeta, x)\omega_\lambda(x). \tag{6.9}$$

where

$$\begin{aligned}
\Omega_i &= d\log(u_1 - v_1) \wedge \ldots \wedge \widehat{d\log(u_i - v_i)} \wedge \ldots \wedge d\log(u_n - v_n) && (6.10)\\
u_i &= \zeta_i + \zeta_i^{-1}, v_i = z_i + z_i^{-1} && (6.11)\\
\Gamma_i &= \{(\zeta \in V_\lambda, |\zeta_j| = 1, j \neq i, |\zeta_i| < 1\} && (6.12)\\
\Sigma_i &= \{(x \in V_\lambda, |x_j| = 1-, j \neq i, |x_i| > 1\}. && (6.13)
\end{aligned}$$

The final formula for $< R_\lambda f, g >$ takes the form

$$\begin{aligned}
< R_\lambda f, g > &= (\tfrac{1}{2\pi i})^n \int_{T^n} \tfrac{f(x)g(x^{-1})}{P_\lambda(x)} dx + \\
&+ (\tfrac{1}{2\pi i})^{2n-1} \textstyle\sum_{i=1}^n \int_{\Sigma_i \times \Gamma_i} f(\zeta)\zeta g(x^{-1}) x^{-1} \Omega_i(\zeta, x)\omega(x).
\end{aligned} \tag{6.14}$$

where Σ_i, Γ_i are the "dual" cycles defined in (6.13), (6.12). Ω_i is defined in (6.10),while $\omega = \omega_\lambda$ is defined after (6.7).

7 Analytic continuation of the resolvent

The integrals involved in (6.14) are well investigated by Poincare, Leray, Griffith among others. On the basis of the corresponding theory one can understand, for instance, how the matrix element $< R_\lambda f, g >$ depends on λ. Roughly speaking if $\lambda \in \mathbf{P}^1_{\mathbf{C}}$, V_λ is a good family of algebraic varieties,ω_λ is a good family of meromorphic differential forms on V_λ,$\gamma_\lambda \in H_*(V_\lambda - Sing\, \omega_\lambda)$ is a cycle no t intersecting the singular set for ω_λ, then the function

$$p(\lambda) = \int_{\gamma_\lambda} \omega_\lambda$$

is holomorphic outside the set $D \subset \mathbf{P}^1_{\mathbf{C}}$ (the discriminant or bifurcation set) which corresponds to varieties $V_\lambda - Sing\, \omega_\lambda$ in nongeneral position.

By applying these arguments to the integrals (6.14) we obtain that the dicriminant locus D in this case consists of $\lambda \in \mathbf{P}^1$ such that on the surfaces $V_\lambda = \{P_\lambda = 0\}$ there arise singular points different from $x = 0$. The final statement is as follows.

Theorem 3 *The function* $\lambda \overset{\varphi}{\mapsto} < R_\lambda f, g >$ *is holomorphic on the universal covering* $\widetilde{\mathbf{P}^1 - D}$, *where* $D = \{\infty, 2k, k \in \mathbf{Z}, k \equiv n(2), |k| \leq n\}$. *Singularities of* φ *in the vicinity of a point* $\lambda \in D$ *takes a form*

$$\sum (z-\lambda)^{\alpha_i} (\log(z-\lambda))^{n_i} c_i(z-\lambda)$$

where the sum is finite, $\alpha_i \in \mathbf{Q}$, n_i *are natural integers, while* $c_i(z)$ *are holomorphic in the neighbourhood of the zero.*

The latter assertion of the theorem is not a special feature of our problem but just express the regularity of the Picard-Fuks differential equations (or the Gauss-Manin connection) and the Landman theorem on quasiunipotence of the monodromy operator.

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Unité de Recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 LE CHESNAY Cedex (France)

Unité de Recherche INRIA Lorraine Technopôle de Nancy-Brabois - Campus Scientifique
615, rue du Jardin Botanique - B.P. 101 - 54602 VILLERS LES NANCY Cedex (France)
Unité de Recherche INRIA Rennes IRISA, Campus Universitaire de Beaulieu 35042 RENNES Cedex (France)
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